

FIRST-ORDER HOMOGENIZATION

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ABSTRACT. We provide a first-order homogenization result for quadratic functionals. In particular, we identify the scaling of the energy and the explicit form of the limiting functional in terms of the first-order correctors. The main novelty of the paper is the use of the dual correspondence between quadratic functionals and PDEs, combined with a refinement of the classical Riemann-Lebesgue Lemma.

1. INTRODUCTION

First-order homogenization does not exist. The non-existence has its roots in two types of boundary effects. The first one arises when the domain is not a disjoint union of suitable rescaled copies of the periodicity cell (see [12, Example 1.12]). The second comes from the oscillatory nature of correctors in the homogenization theory (see [2, Equation (2.12)]). Nevertheless, in this manuscript, we will provide a first-order Γ -convergence result for quadratic energies under suitable assumptions which allow us to ‘forget’ about the boundary. In particular, the scaling and the principal part of the energy in the bulk will be identified.

Nowadays, homogenization is a very well-established mathematical theory describing how the microstructure affects the overall behavior of a material (see, e.g., [8, 10]). The mathematical literature on the topic is too vast for an exhaustive list, and thus we limit ourselves to mention here some examples where it has been successfully applied: from thin structures (see, for instance, [6, 11, 14, 29]), to phase separation (see, for instance, [16, 17, 18]), and from micromagnetism (see [3, 15, 20, 21]) to supremal functionals (see [13, 22]).

The prototypical example is a family of functionals $\mathcal{F}_\varepsilon : L^p(\Omega; \mathbb{R}^M) \rightarrow \mathbb{R} \cup \{+\infty\}$, for $p \in (1, \infty)$, with $\varepsilon > 0$ being the length scale characterizing the fine structure, of the form

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_{\Omega} W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx & \text{if } u \in W_0^{1,p}(\Omega; \mathbb{R}^M), \\ +\infty & \text{else.} \end{cases}$$

The first variable of the energy density W accounts for the presence of a periodic microstructure, which is reflected in requiring that $W(\cdot, \xi)$ is a periodic function. The variational investigation of the periodic homogenization goes back to the end of Seventies. In [26], the limiting functional $\mathcal{F}_{\text{hom}}$ of $\mathcal{F}_\varepsilon$ has been fully characterized in the scalar case, i.e. $M = 1$, and under the assumptions of convexity and of p -growth of $W(x, \cdot)$. The Γ -limit $\mathcal{F}_{\text{hom}} : L^p(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is given by

$$\mathcal{F}_{\text{hom}}(u) := \begin{cases} \int_{\Omega} W_{\text{hom}}(\nabla u(x)) dx & \text{if } u \in W_0^{1,p}(\Omega), \\ +\infty & \text{else,} \end{cases} \quad (1.1)$$

with the effective energy density $W_{\text{hom}} : \mathbb{R}^N \rightarrow \mathbb{R}$ being characterized through the so-called *cell formula*

$$W_{\text{hom}}(\xi) := \inf \left\{ \int_{[0,1]^N} W(y, \xi + \nabla u(x)) : u \in W_0^{1,p}([0,1]^N) \right\}. \quad (1.2)$$

Removing the assumption of convexity and in the vectorial framework, the analysis of the Γ -limit has been carried out independently in [9] and [27]. In this case, the limiting energy $\mathcal{F}_{\text{hom}} : L^p(\Omega; \mathbb{R}^M) \rightarrow \mathbb{R} \cup \{+\infty\}$

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is again of the form (1.1) but the homogenized energy density $W_{\text{hom}} : \mathbb{R}^{M \times N} \rightarrow \mathbb{R}$ is characterized by the *asymptotic cell formula*

$$W_{\text{hom}}(\Xi) := \lim_{k \rightarrow \infty} \frac{1}{k^N} \inf \left\{ \int_{[0,k)^N} W(y, \Xi + \nabla u(x)) : u \in W_0^{1,p}([0,k)^N; \mathbb{R}^M) \right\}. \quad (1.3)$$

It is worth noticing that in the scalar setting and assuming the convexity of W in the second variable, formula (1.3) turns into (1.2) (see [27, Lemma 4.1]).

The aim of the present paper is to undertake a first-order analysis of a suitable version functional $\mathcal{F}_\varepsilon$ via Γ -convergence. We focus on quadratic energies $F_\varepsilon : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ of the form

$$F_\varepsilon(u) := \begin{cases} \int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u(x) \cdot \nabla u(x) \, dx - \int_{\Omega} f(x)u(x) \, dx & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{else,} \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $f \in L^2(\Omega)$ and $A : \Omega \rightarrow \mathbb{R}^{N \times N}$ is a matrix-valued function in L^∞ that is $[0,1)^N$ -periodic, symmetric and with lower and upper quadratic bounds. To identify the contribution of the bulk in the first-order limit, we mimic the approach deployed by Allaire and Amar in [2] assuming that

$$\Omega := [0,1)^N \quad \text{and} \quad \varepsilon = \frac{1}{n}, \quad \text{with } n \in \mathbb{N} \setminus \{0\}.$$

This allows us to get rid of the first type of boundary effect. To tackle the second issue with boundary effect, we restrict the admissible class for the source term f (see Section 2.2 for further details). Using the asymptotic expansion of functionals given in [4] in terms of Γ -convergence, we consider the functionals $F_\varepsilon^1 : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as

$$F_\varepsilon^1(u) := \frac{F_\varepsilon(u) - \min_{H_0^1(\Omega)} F_{\text{hom}}}{\varepsilon}.$$

The principal result of the present manuscript is the identification of the scale ε above as well as the Γ -limit of F_ε^1 (see Theorem 2.7). The main novelty lies in the use of the dual correspondence between quadratic functionals and PDEs. To the best of the authors' knowledge, this is the first time that these two theories are combined together to get a variational result.

We briefly outline the strategy we employ. The unique minimizer of F_ε turns out to be the unique solution of the following elliptic problem with Dirichlet boundary conditions

$$\begin{cases} -\operatorname{div} \left(A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) \right) = f(x) & \text{in } \Omega, \\ u_\varepsilon(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

The investigation of such an elliptic problem has been broadly carried out by many authors, [1, 7, 25] to name a few (see [8] for an extensive review on the topic). To get a homogenized equation, the classical strategy relies on the two-scale expansion developed in [1, 28] of the solution u_ε :

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \dots, \quad (1.4)$$

where u_0 is the solution of the homogenized equation given by

$$\begin{cases} -\operatorname{div} (A_{\text{hom}} \nabla u_0(x)) = f(x) & \text{in } \Omega, \\ u_0(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where A_{hom} is defined through the cell formula (1.2) with $W(y, \xi + \nabla u) = A(y)(\xi + \nabla u(x)) \cdot (\xi + \nabla u(x))$. Moreover, the function u_1 is defined through the first-order correctors (see Section 3 for the precise definition). One would be tempted to use the ansatz (1.4) in the variational analysis for the functional F_ε^1 to deduce the limiting energy. However, this idea does not work out. The reason is that the following estimate

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}$$

turns out to be sharp (see, e.g., [7]). This surprising result suggests the presence of another phenomenon, known as *boundary layers*. They are further first-order corrections needed to match the boundary conditions (see [2, 5, 24]). Due to the high oscillatory nature of these functions, their energy contribution is not clearly quantifiable with respect to the parameter ε . This is what we have referred to as the second type of boundary effect. In order to avoid this high oscillatory behavior at the boundary, we essentially consider the case where the function u_ε is compactly supported in Ω by requiring the source term f to be in a specific class (see Assumption (H5) in Section 2.2). This enables us to get the first-order Γ -limit by using the ansatz in (1.4) together with a refinement of the classical Riemann-Lebesgue Lemma (see Proposition 4.3).

Finally, we show that the first-order Γ -convergence analysis is not needed when the functional F_ε only depends on the function u and not on its gradient ∇u (see Theorem 2.12). Indeed, in such a case, we have that the value of the minimum of the functional F_ε is the same as the one of F_{hom} . This implies that the expansion by Γ -convergence does not provide additional information on the minimizers of the functional F_ε .

The paper is organized as follows. In Section 2 we specify the set-up of the problem. The preliminaries and the technical results are given in Section 3 and Section 4, respectively. We then turn to the proofs of the main result: in Section 5 we provide the compactness, while Section 6 and Section 7 are devoted to the lower and upper bound, respectively. Finally, Section 8 is devoted to the proof of the first-order homogenization for functionals in L^p , for $p \in (1, \infty)$.

2. SET-UP OF THE PROBLEM AND MAIN RESULT

2.1. Basic notation. Here we collect the basic notation we are going to use throughout the manuscript. Let $Y \subset \mathbb{R}^N$ be a periodicity cell, namely

$$Y = \left\{ \sum_{i=1}^N \lambda_i v_i : 0 < \lambda_i < 1, \sum_{i=1}^N \lambda_i = 1 \right\},$$

where $v_1, \dots, v_N$ is a basis of $\mathbb{R}^N$. Without loss of generality, up to a translation, we can even assume that Y has its baricenter at the origin. This assumption is just to simplify the writing of the main result. The space $H_{\text{per}}^1(Y)$ is the subset of $H^1(Y)$ of functions with periodic boundary conditions. More precisely, $u \in H^1(Y)$ if and only if the function $\tilde{u} : \mathbb{R}^N \rightarrow \mathbb{R}$ defined as $\tilde{u}(y) := u(\tilde{y})$ belongs to $H_{\text{loc}}^1(\mathbb{R}^N)$, where

$$y = \sum_{i=1}^N \lambda_i v_i, \quad \tilde{y} := \sum_{i=1}^N \{\lambda_i\} v_i,$$

and $\{\lambda_i\} := \lambda_i - \lfloor \lambda_i \rfloor$.

Given a function f , the notation f^ε stands for $f^\varepsilon(x) := f(x/\varepsilon)$. Moreover, we denote by ∂_i the i^{th} partial derivative operator with respect to the variable x , and by ∂_{y_i} the i^{th} partial derivative operator with respect to the variable y . In particular, we have that

$$\partial_i f^\varepsilon(x) = \frac{1}{\varepsilon} \partial_{y_i} f^\varepsilon(x).$$

Finally, the symbol $\langle \cdot \rangle_Y$ denotes the average over Y , i.e.,

$$\langle f \rangle_Y := \frac{1}{|Y|} \int_Y f(y) dy,$$

with $|Y|$ being the N -dimensional Lebesgue measure of Y .

2.2. Main result. Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, and let $f \in L^2(\Omega)$. Let $A : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times N}$ be a matrix-valued function in L^∞ such that

- (H1) A is Y -periodic;
- (H2) A is symmetric, i.e., $a_{ij}(y) = a_{ji}(y)$;

(H3) there exist two positive constants α, β such that

$$\alpha|\xi|^2 \leq A(y)\xi \cdot \xi \leq \beta|\xi|^2,$$

for all $\xi \in \mathbb{R}^N$.

For $\varepsilon > 0$, let $F_\varepsilon : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ be the functional defined as

$$F_\varepsilon(u) := \int_{\Omega} A^\varepsilon(x) \nabla u(x) \cdot \nabla u(x) \, dx - \int_{\Omega} f(x)u(x) \, dx,$$

if $u \in H_0^1(\Omega)$, and as $F_\varepsilon(u) := +\infty$ otherwise in $L^2(\Omega)$.

Under Assumptions (H1)-(H3), we know (see, for instance, [26], [27, Theorem 1.3], or [19, Corollary 24.5]) that $\{F_\varepsilon\}_\varepsilon$ Γ -converges with respect to the weak topology of $H^1(\Omega)$, or equivalently the strong topology of $L^2(\Omega)$, to the effective functional $F_{\text{hom}}^0 : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$F_{\text{hom}}^0(u) := \int_{\Omega} A_{\text{hom}} \nabla u(x) \cdot \nabla u(x) \, dx - \int_{\Omega} f(x)u(x) \, dx,$$

if $u \in H_0^1(\Omega)$, and by $F_{\text{hom}}^0(u) := +\infty$ otherwise in $L^2(\Omega)$. Here, the effective matrix A_{hom} is a constant matrix given by the *cell-formula*

$$A_{\text{hom}}\xi \cdot \xi := \inf \left\{ \int_Y A(y)(\xi + \nabla \phi(y)) \cdot (\xi + \nabla \phi(y)) \, dy : \phi \in H_{\text{per}}^1(Y) \right\}. \quad (2.1)$$

We refer to this Γ -convergence result as the zeroth-order term in the expansion by Γ -convergence of F_ε . For our analysis, we need to recall the following. Using the fact that the functional in (2.1) is quadratic, for each $\xi \in \mathbb{R}^N$, the minimization problem defining $A_{\text{hom}}\xi \cdot \xi$ has a unique solution (up to an additive constant), denoted by ψ_ξ .

Definition 2.1. Let $e_1, \dots, e_N$ be the standard orthonormal basis of $\mathbb{R}^N$. For each $i = 1, \dots, N$, let $\psi_i \in H_{\text{per}}^1(Y)$ be the unique solution to

$$\inf \left\{ \int_Y A(y)(e_i + \nabla \phi(y)) \cdot (e_i + \nabla \phi(y)) \, dy : \phi \in H_{\text{per}}^1(Y), \int_Y \phi(y) \, dy = 0 \right\}.$$

The function ψ_i is called the first-order corrector for A associated to the vector e_i .

Remark 2.2. It turns out that the map $\xi \mapsto \psi_\xi$ is linear (see [19, Example 25.5]). Namely,

$$\psi_\xi = \sum_{i=1}^N \psi_i \xi_i,$$

where $\xi = (\xi_1, \dots, \xi_N)$. Therefore, the knowledge of the first-order correctors for A is sufficient to obtain A_{hom} .

The goal of this paper is to develop further the expansion by Γ -convergence of F_ε (see [4] for further details). As explained in the Introduction, there are two issues with boundary effects in obtaining such a result. Thus, in order to carry out our analysis, we need to assume the following

(H4) $\Omega := Y$ and we take the sequence $\varepsilon_n = \frac{1}{n}$, with $n \in \mathbb{N} \setminus \{0\}$;

(H5) It holds that

$$f = -\text{div}(A_{\text{hom}} \nabla g),$$

for some $g \in C_c^\infty(Y)$.

Assumption (H4) is to ensure that Ω can be obtained as a disjoint union of copies of rescaled versions of the periodicity cell Y . On the other hand, assumption (H5) is in order to avoid using boundary layers, which have two main issues: they do not have a variational definition, and their contribution to the energy is not clearly quantifiable in terms of the parameter ε , which prevents us from getting an order of the energy. From now on, recalling assumption (H4), we will denote by F_n the functional F_{ε_n} .

We are now in position to write the asymptotic expansion through Γ -convergence we will study.

Definition 2.3. For $n \in \mathbb{N} \setminus \{0\}$, we define the functional $F_n^1 : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ as

$$F_n^1(u) := \frac{F_n(u) - \min_{H_0^1(\Omega)} F_{\text{hom}}^0}{\varepsilon_n}. \quad (2.2)$$

Remark 2.4. Note that, using standard estimates (see the proof of Proposition 5.1), it is possible to prove that, for each $n \in \mathbb{N} \setminus \{0\}$, the minimization problem

$$\min_{u \in H_0^1(\Omega)} F_n(u)$$

admits a unique solution $u_n^{\min} \in H^1(\Omega)$. In a similar way, it is possible to prove that the minimization problem

$$\min_{u \in H_0^1(\Omega)} F_{\text{hom}}^0(u)$$

admits a unique minimizer $u_0^{\min} \in H^1(\Omega)$. In particular, we have that

$$F_n^1(u) = \frac{F_n(u) - F_{\text{hom}}^0(u_0^{\min})}{\varepsilon_n}.$$

Moreover, the Γ -convergence of F_n to F_{hom}^0 together with the compactness of $\{u_n^{\min}\}_n$ yields that $u_n^{\min} \rightarrow u_0^{\min}$ in $L^2(\Omega)$ as $n \rightarrow \infty$.

Remark 2.5. Note that assumption (H5) implies that $u_0^{\min} = g$. In particular, $u_0^{\min} \in C_c^\infty(\Omega)$.

Our goal is to study the Γ -limit of the family of functionals $\{F_n^1\}_n$. To this end, we introduce the candidate limiting functional.

Definition 2.6. Define the functional $F_{\text{hom}}^1 : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ as

$$\begin{aligned} F_{\text{hom}}^1(u) := & \sum_{i,j=1}^N \int_{\Omega} \nabla(\partial_i u_0^{\min} \partial_j u_0^{\min})(x) \, dx \cdot \langle y [a_{ij} + 2Ae_j \cdot \nabla \psi_i + A \nabla \psi_i \cdot \nabla \psi_j] \rangle_Y \\ & + 2 \sum_{i=1}^N \int_{\Omega} \langle \psi_i A \rangle_Y \nabla u_0^{\min}(x) \cdot \partial_i u_0^{\min}(x) \, dx \\ & + 2 \sum_{j=1}^N \sum_{i=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \langle \psi_i A \nabla_y \psi_j \rangle_Y \cdot \partial_i \nabla u_0^{\min}(x) \, dx \\ & - \sum_{j=1}^N \langle \psi_j \rangle_Y \int_{\Omega} f(x) \partial_j u_0^{\min}(x) \, dx \end{aligned} \quad (2.3)$$

if $u = u_0^{\min}$, and $+\infty$ else in $L^2(\Omega)$. Here, ψ_i are the first-order correctors defined in Definition 2.1.

We now state the main result of the present paper.

Theorem 2.7. Let F_n^1 and F_{hom}^1 be the functionals given by (2.2) and (2.3), respectively. Assume that Assumptions (H1)-(H5) hold. Then, we have the following:

- (i) Let $\{u_n\}_n$ be a sequence in $H^1(\Omega)$ such that $\sup_n F_n^1(u_n) < \infty$. Then, $\{u_n\}_n$ converges in $L^2(\Omega)$ to $u_0^{\min}$, where $u_0^{\min}$ is the unique minimizer of F_{hom}^0 .
- (ii) The sequence $\{F_n^1\}_n$ Γ -converges with respect to $L^2(\Omega)$ topology to F_{hom}^1 .

It is worth noticing that the above result can be read as follows: in the bulk, the order of the energy F_n is ε_n , and the normalized difference from $\min F_{\text{hom}}^0$ is given by the (constant) functional F_{hom}^1 .

We show that the analogous first-order Γ -expansion is trivial in the case of functionals defined on L^p . We are able to prove this statement in a more general setting than that considered above. Fix $p \in (1, \infty)$ and $M \geq 1$. Let $V : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}$ be a Carathéodory function such that

- (A1) For each $z \in \mathbb{R}^M$, the function $x \mapsto V(x, z)$ is Y -periodic;

(A2) There exists $0 < c_1 < c_2 < +\infty$ such that

$$c_1(|z|^p - 1) \leq V(x, z) \leq c_2(|z|^p + 1),$$

for all $x \in \Omega$ and $z \in \mathbb{R}^M$.

Note that in this case the source term would only be a part of the function V . That is why there is no analogue of assumption (H5).

Definition 2.8. For $n \in \mathbb{N} \setminus \{0\}$, define $G_n : L^p(\Omega; \mathbb{R}^M) \rightarrow \mathbb{R} \cup \{+\infty\}$ as

$$G_n(u) := \int_{\Omega} V\left(\frac{x}{\varepsilon_n}, u(x)\right) dx.$$

Remark 2.9. Note that there is no loss of generality in assuming the function V to be convex in the second variable. Indeed, the relaxation of G_n with respect to the weak- $L^p(\Omega)$ topology is given by

$$\overline{G}_n(u) = \int_{\Omega} V^c\left(\frac{x}{\varepsilon_n}, u(x)\right) dx$$

where V^c denotes the convex envelope of the function V in the second variable.

We now introduce the homogenized functional.

Definition 2.10. For $z \in \mathbb{R}^M$, let

$$V_{\text{hom}}(z) := \inf \left\{ \frac{1}{|Y|} \int_Y V^c(y, z + \varphi(y)) dy : \varphi \in L^p(Y; \mathbb{R}^M), \int_Y \varphi(y) dy = 0 \right\}.$$

Define the functional $G_{\text{hom}} : L^p(\Omega; \mathbb{R}^M) \rightarrow \mathbb{R} \cup \{+\infty\}$ as

$$G_{\text{hom}} := \int_{\Omega} V_{\text{hom}}(u(x)) dx.$$

Using the same arguments as in the proof of [17, Theorem 3.3], we get the following.

Lemma 2.11. Assume that Assumptions (A1)-(A2) hold. Then, $G_n \rightarrow G_{\text{hom}}$ with respect to the weak- $L^p(\Omega)$ topology.

Next result justifies our claim that the Γ -expansion is trivial in such a case.

Theorem 2.12. Assume that Assumptions (A1)-(A2), as well as (H4) hold and that, for all $y \in Y$, the function $u \mapsto V(y, u)$ is strictly convex. Let $m \in \mathbb{R}^M$. Then,

$$\min \left\{ G_n(v) : v \in L^p(\Omega; \mathbb{R}^M), \int_{\Omega} u(x) dx = m \right\} = \min \left\{ G_{\text{hom}}(v) : v \in L^p(\Omega; \mathbb{R}^M), \int_{\Omega} u(x) dx = m \right\},$$

for all $n \in \mathbb{N} \setminus \{0\}$.

Remark 2.13. The assumption of strict convexity is needed only to simplify the strategy of the proof, since we need to invert the relation

$$\partial_u V(x, v) = c.$$

Strict convexity gives us a unique inverse, while with convexity alone we would have to get a slightly more involved argument. Since this result is to show that for these functionals there is no need to a first order Γ -expansion, we decided to use this extra technical assumption.

Remark 2.14. Note that, in this case, boundary conditions are not natural. This is why we consider a mass constraint instead.

3. PRELIMINARIES

3.1. Recall of homogenization. We recall the foundations of the homogenization theory for elliptic equations. Even if this theory is classical by now, we revisit it since we will use a slightly different definition of the correctors than that in [2, 7].

Let Ω be a bounded and open subset of $\mathbb{R}^N$. Let A be the matrix-valued function satisfying Assumptions (H1)-(H3). For a given $f \in L^2(\Omega)$, we consider the following equation

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x)\nabla u_\varepsilon(x)) = f(x) & \text{in } \Omega, \\ u_\varepsilon(x) = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

It is well-known that this problem admits a unique solution in $H_0^1(\Omega)$ (see, e.g., [7]). To carry out a homogenization procedure, the solution u_ε is assumed to have the following two-scale expansion

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots,$$

where each u_i is Y -periodic with respect to the fast variable $y = \frac{x}{\varepsilon}$. The variables x and y are treated as independent. Plugging such an expansion into (3.1) and identifying powers of ε , we get a cascade of equations. Here, we only care about the second-order expansion. Therefore, setting $\mathcal{A}_\varepsilon \phi(x) := -\operatorname{div}(A^\varepsilon(x)\nabla \phi(x))$, we deduce that

$$\mathcal{A}_\varepsilon = \varepsilon^{-2} \mathcal{A}_0 + \varepsilon^{-1} \mathcal{A}_1 + \mathcal{A}_2,$$

where

$$\begin{aligned} \mathcal{A}_0 &:= -\sum_{i=1}^N \partial_{y_i} \left(\sum_{j=1}^N a_{ij}(y) \partial_{y_j} \right), \\ \mathcal{A}_1 &:= -\sum_{i=1}^N \partial_{y_i} \left(\sum_{j=1}^N a_{ij}(y) \partial_j \right) - \sum_{i=1}^N \partial_i \left(\sum_{j=1}^N a_{ij}(y) \partial_{y_j} \right), \\ \mathcal{A}_2 &:= -\sum_{i=1}^N \partial_i \left(\sum_{j=1}^N a_{ij}(y) \partial_j \right). \end{aligned}$$

Using (3.1), matching powers of ε up to second order gives us the following equations

$$\mathcal{A}_0 u_0 = 0, \quad (3.2)$$

$$\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0 = 0, \quad (3.3)$$

$$\mathcal{A}_0 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_2 u_0 = f, \quad (3.4)$$

$$\mathcal{A}_0 u_3 + \mathcal{A}_1 u_3 + \mathcal{A}_2 u_1 = 0. \quad (3.5)$$

From (3.2), it follows that $u_0(x, y) \equiv u_0(x)$. The solution u_1 to (3.3) is given by

$$u_1\left(x, \frac{x}{\varepsilon}\right) = \sum_{j=1}^N \psi_j^\varepsilon(x) \partial_j u_0(x) + \tilde{u}_1(x), \quad (3.6)$$

where, for $j = 1, \dots, N$, ψ_j is the unique solution in $H_{\text{per}}^1(Y)$ to the problem

$$\begin{cases} \mathcal{A}_0 \psi_j(y) = \sum_{i=1}^N \partial_{y_i} a_{ij}(y) & \text{in } Y, \\ \int_Y \psi_j(y) dy = 0, \\ y \mapsto \psi_j(y) & Y\text{-periodic.} \end{cases} \quad (3.7)$$

Namely, for $j = 1, \dots, N$, the function ψ_j are the first-order corrector defined in Definition 2.1. The solution to equation (3.4) is given by

$$u_2\left(x, \frac{x}{\varepsilon}\right) = \sum_{i,j=1}^N \chi_{ij}^\varepsilon(x) \partial_{ij}^2 u_0(x) + \sum_{j=1}^N \psi_j^\varepsilon(x) \partial_j \tilde{u}_1(x) + \tilde{u}_2(x), \quad (3.8)$$

where for all $i, j = 1, \dots, n$, the *second-order corrector* $\chi_{ij} \in H_{\text{per}}^1(Y)$ satisfies the following auxiliary problem

$$\begin{cases} \mathcal{A}_0 \chi_{ij}(y) = b_{ij}(y) - \int_Y b_{ij}(y) dy & \text{in } Y, \\ \int_Y \chi_{ij}(y) dy = 0, \\ y \mapsto \chi_{ij}(y) & Y\text{-periodic,} \end{cases} \quad (3.9)$$

with

$$b_{ij}(y) := a_{ij}(y) + \sum_{k=1}^N a_{ik}(y) \partial_{y_k} \psi_j(y) + \sum_{k=1}^N \partial_{y_k} (a_{ki} \psi_j)(y). \quad (3.10)$$

It is worth recalling that the function $\tilde{u}_1$ in (3.6) can be taken identically equal to zero if one is only interested in the first-order expansion of u_ε . Otherwise, $\tilde{u}_1$ is determined by the compatibility condition of equation (3.5), namely,

$$\text{div}(A_{\text{hom}} \nabla \tilde{u}_1(x)) = - \sum_{i,j,k=1}^N c_{ijk} \partial_{ijk}^3 u_0^{\min}(x), \quad (3.11)$$

where A_{hom} is the homogenized matrix defined by (2.1) and for $i, j, k = 1, \dots, N$, the constant c_{ijk} is defined as

$$c_{ijk} := \left\langle \sum_{l=1}^N a_{kl} \partial_{y_l} \chi_{ij} + a_{ij} \psi_k \right\rangle_Y. \quad (3.12)$$

Since in the present paper, we are interested in expanding u_ε up to the second-order, the function $\tilde{u}_2$ in (3.8) can be taken identically equal to zero.

Remark 3.1. In the following, we will use the above theory for the function $u_0^{\min}$ in place of u_0 .

3.2. Periodic functions. We collect some useful properties of the space $L_{\text{per}}^2(Y; \mathbb{R}^N)$ of the periodic functions (see [25, Chapter 1] for further details).

The space $L_{\text{sol}}^2(Y)$ of solenoidal periodic functions is defined as

$$L_{\text{sol}}^2(Y) := \{f \in L_{\text{per}}^2(Y; \mathbb{R}^N) : \text{div} f = 0 \text{ in } \mathbb{R}^N\},$$

where the equality $\text{div} f = 0$ is to be intended in distributional sense, i.e.,

$$\int_{\mathbb{R}^N} f(x) \cdot \nabla \varphi(x) dx = 0, \quad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^N).$$

The space $L_{\text{sol}}^2(Y)$ turns out to be a closed subspace of $L^2(Y; \mathbb{R}^N)$. Setting

$$\mathcal{V}_{\text{pot}}^2(Y) := \{\nabla u : u \in H_{\text{per}}^1(Y; \mathbb{R}^N)\},$$

we immediately deduce the following orthogonal representation

$$L_{\text{per}}^2(Y; \mathbb{R}^N) = L_{\text{sol}}^2(Y) \oplus \mathcal{V}_{\text{pot}}^2(Y).$$

The next proposition provides a representation of solenoidal functions, which will be a key tool in Proposition 4.9.

Proposition 3.2. *Let $f \in L_{\text{sol}}^2(Y)$. Then, $f = (f_1, \dots, f_N)$ can be represented in the form*

$$f_j(x) = \langle f_j \rangle_Y + \sum_{i=1}^N \partial_i \alpha_{ij}(x), \quad \text{for all } j = 1, \dots, N,$$

where $\alpha_{ij} \in H^1(Y)$ is such that $\alpha_{ij} = -\alpha_{ji}$ and $\langle \alpha_{ij} \rangle_Y = 0$, for all $i, j = 1, \dots, N$.

Proof. Without loss of generality, we can assume that $Y = [0, 2\pi)^N$. Using the Fourier series of f , we have that

$$f = \langle f \rangle_Y + \sum_{\substack{k \in \mathbb{Z}^N \\ k \neq 0}} f^k e^{ik \cdot x},$$

where f^k is the Fourier coefficient of f . We claim that $f^k \cdot k = 0$, for each $k \in \mathbb{Z}^N$. Indeed, for fixed $k \in \mathbb{Z}^N \setminus \{0\}$, we can decompose f^k as $f^k = C_1^k k + C_2^k v$, for some constants C_1^k, C_2^k and $v \in (\text{span}(k))^\perp$. Therefore,

$$f = \langle f \rangle_Y + \sum_{\substack{k \in \mathbb{Z}^N \\ k \neq 0}} C_1^k k e^{ik \cdot x} + \sum_{\substack{k \in \mathbb{Z}^N \\ k \neq 0}} C_2^k v e^{ik \cdot x}.$$

Noticing that $ik e^{ik \cdot x} = \nabla(e^{ik \cdot x}) \in \mathcal{V}_{\text{pot}}^2(Y)$ and since by assumption, we know that $f \in L_{\text{sol}}^2(Y)$, we obtain that $C_1^k = 0$, for any $k \in \mathbb{Z}^N \setminus \{0\}$. This leads us to conclude that $f^k \cdot k = 0$, and that $C_2^k v = f^k$. Therefore,

$$f^k = f^k - \frac{f^k \cdot k}{|k|^2} k,$$

which writes, component-wise, as

$$f_j^k = \sum_{i=1}^N g_{ij}^k k_i, \quad \text{with} \quad g_{ij} := \frac{f_j^k k_i - f_i^k k_j}{|k|^2}.$$

Therefore, for each $j = 1, \dots, N$, we get that

$$f_j = \langle f_j \rangle_Y + \sum_{\substack{k \in \mathbb{Z}^N \\ k \neq 0}} f_j^k e^{ik \cdot x} = \langle f_j \rangle_Y + \sum_{\substack{k \in \mathbb{Z}^N \\ k \neq 0}} \sum_{i=1}^N g_{ij}^k k_i e^{ik \cdot x} = \langle f_j \rangle_Y + \sum_{i=1}^N \sum_{\substack{k \in \mathbb{Z}^N \\ k \neq 0}} g_{ij}^k k_i e^{ik \cdot x}.$$

Thus, by defining

$$\alpha_{ij}(x) := -i \sum_{\substack{k \in \mathbb{Z}^N \\ k \neq 0}} g_{ij}^k e^{ik \cdot x},$$

we get the desired result. $\square$

4. TECHNICAL LEMMATA

4.1. First order Riemann-Lebesgue Lemma. We prove a quantitative version of the Riemann-Lebesgue Lemma. Since we will use this latter result several times, we recall it in here for the reader's convenience.

Lemma 4.1 (Riemann-Lebesgue Lemma). *Let $p \in [1, \infty)$, and let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let $g \in L^p(\mathbb{R}^N)$ be an Y -periodic function. Then,*

$$\lim_{n \rightarrow \infty} \int_{\Omega} g(nx) \varphi(x) dx = \langle g \rangle_Y \int_{\Omega} \varphi(x) dx,$$

for all $\varphi \in L^{p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 4.2. The above result holds also for $p = +\infty$, but in that case we need the test function $\varphi \in L^1(\Omega)$.

The refined result requires more stringent assumptions. Nevertheless, we will later show how to relax some of them.

Proposition 4.3 (First order Riemann-Lebesgue Lemma). *Let $p \in [1, \infty)$, and let $g \in L^p(\mathbb{R}^N)$ be an Y -periodic function. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set such that, up to a set of Lebesgue measure zero, can be written as*

$$\Omega = \bigcup_{i=1}^k (x_i + Y), \quad x_i \in \mathbb{Z}^N, \quad (x_i + Y) \cap (x_j + Y) = \emptyset \text{ if } i \neq j. \quad (4.1)$$

Then,

$$\lim_{n \rightarrow \infty} n \left[\int_{\Omega} g(nx) \varphi(x) dx - \langle g \rangle_Y \int_{\Omega} \varphi(x) dx \right] = \int_{\Omega} \nabla \varphi(x) dx \cdot [\langle yg \rangle_Y - \langle y \rangle_Y \langle g \rangle_Y],$$

for all $\varphi \in C^2(\Omega) \cap C(\bar{\Omega})$.

Proof. Let $\{z_i\}_i$ be an enumeration of $\mathbb{Z}^N$. For each $n \in \mathbb{N} \setminus \{0\}$, let

$$I_n := \left\{ i \in \mathbb{N} : \frac{1}{n}(z_i + Y) \subset \Omega \right\}.$$

Note that, by (4.1), we get that (up to a set of Lebesgue measure zero)

$$\Omega = \bigcup_{i \in I_n} \frac{1}{n}(z_i + Y)$$

being a disjoint union. Moreover, the cardinality of I_n is $n^N |\Omega|/|Y|$. Recall that, by periodicity of g , we have that $g(z_i + y) = g(y)$ for all $z_i \in \mathbb{Z}^N$. Then,

$$\begin{aligned} & \int_{\Omega} g(nx) \varphi(x) dx - \frac{1}{|Y|} \int_Y g(y) dy \int_{\Omega} \varphi(x) dx \\ &= \sum_{i \in I_n} \frac{1}{n^N} \left[\int_Y g(z_i + y) \varphi\left(\frac{z_i + y}{n}\right) dy - \frac{1}{|Y|} \int_Y \varphi\left(\frac{z_i + y}{n}\right) dy \int_Y g(y) dy \right] \\ &= \sum_{i \in I_n} \frac{1}{n^N} \left[\int_Y g(y) \left[\varphi\left(\frac{z_i + y}{n}\right) - \varphi\left(\frac{z_i}{n}\right) \right] dy - \frac{1}{|Y|} \int_Y \left[\varphi\left(\frac{z_i + y}{n}\right) - \varphi\left(\frac{z_i}{n}\right) \right] dy \int_Y g(y) dy \right]. \end{aligned}$$

We now use a second-order Taylor expansion for φ to write

$$\varphi\left(\frac{z_i + y}{n}\right) - \varphi\left(\frac{z_i}{n}\right) = \frac{1}{n} \nabla \varphi\left(\frac{z_i}{n}\right) \cdot y + \frac{1}{2n^2} H\varphi\left(\frac{z_i}{n}\right) [y, y] + o\left(\frac{1}{n^2}\right),$$

where $H\varphi(x)$ denotes the Hessian matrix of φ at x . Thus, we get that

$$\begin{aligned} & \int_{\Omega} g(nx) \varphi(x) dx - \frac{1}{|Y|} \int_Y g(y) dy \int_{\Omega} \varphi(x) dx \\ &= \frac{1}{n} \sum_{i \in I_n} \frac{1}{n^N} \nabla \varphi\left(\frac{z_i}{n}\right) \cdot \left[\int_Y g(y) y dy - \frac{1}{|Y|} \int_Y y dy \int_Y g(y) dy \right] + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Sending $n \rightarrow \infty$, and noting that

$$\sum_{i \in I_n} |Y| \frac{1}{n^N} \nabla \varphi\left(\frac{z_i}{n}\right)$$

is the Riemann sum for the integral of $\nabla \varphi$ over Ω , gives the desired result. $\square$

Remark 4.4. Note that the above result is invariant under translation of the periodicity grid and of the function g . In particular, there is no loss of generality in assuming that $\langle y \rangle_Y = 0$, namely that the barycenter of the periodicity cell is at the origin.

Remark 4.5. In particular, the above result implies that

$$\lim_{n \rightarrow \infty} n \left[\int_{\Omega} g(nx) \varphi(x) dx - \langle g \rangle_Y \int_{\Omega} \varphi(x) dx \right] = 0,$$

for each Y -periodic function $\varphi \in C^2(\mathbb{R}^N)$.

Finally, we present some extensions of the above result. Their proofs follow the same argument used above, and therefore we will not repeat them in here.

The first extension uses the fact that, in dimension $N \geq 3$, given a general open set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary (or even whose boundary has finite $\mathcal{H}^{N-1}$ measure), the contribution of the cubes $(z_i + Y)/n$ that intersect the boundary of Ω is of order $n^{1-N} \ll n^{-1}$.

Proposition 4.6. *Let $N \geq 3$, and let $\Omega \subset \mathbb{R}^N$ be an open set with Lipschitz boundary. Let $p \in [1, \infty)$, and let $g \in L^p(\mathbb{R}^N)$ be an Y -periodic function. Then,*

$$\lim_{n \rightarrow \infty} n \left[\int_{\Omega} g(nx) \varphi(x) dx - \langle g \rangle_Y \int_{\Omega} \varphi(x) dx \right] = \int_{\Omega} \nabla \varphi(x) dx \cdot [\langle yg \rangle_Y - \langle y \rangle_Y \langle g \rangle_Y],$$

for all $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$.

Remark 4.7. Note that, in dimension $N = 2$, the contribution of the terms at the boundary of Ω is of the order n^{-1} . It is therefore, in general, not possible to estimate this term.

Finally, we can extend the above result to the nonlinear case.

Proposition 4.8. *Let $p \in [1, \infty)$. Let $V : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function that is Y -periodic in the second variable, and such that $t \mapsto V(x, t)$ is of class C^2 for all $x \in \mathbb{R}^N$. Assume that (4.1) hold. Then,*

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left[\int_{\Omega} V(nx, \varphi(x)) dx - \int_{\Omega} \langle V(\cdot, \varphi(x)) \rangle_Y dx \right] \\ = \int_{\Omega} \frac{1}{|Y|} \int_Y \partial_t V(y, \varphi(x)) \nabla \varphi(x) \cdot y dy dx - \frac{1}{|Y|^2} \int_{\Omega} \int_Y \int_Y \partial_t V(z, \varphi(x)) \nabla \varphi(x) \cdot y dz dy dx, \end{aligned}$$

for all $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$.

4.2. Estimates. In this section we prove the fundamental estimate that allows us to consider the ‘surrogate’ sequence given by the expansion with first and second-order correctors in place of the minimizer $u_{\varepsilon}^{\min}$. Recall that $u_0^{\min} \in C_c^{\infty}(\Omega)$ (see Remark 2.5).

Proposition 4.9. *Let $u_{\varepsilon}^{(2)}$ be the function defined as*

$$u_{\varepsilon}^{(2)}(x, y) := u_0^{\min}(x) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y), \quad (4.2)$$

with u_1 and u_2 being defined as in (3.6) and (3.8), respectively. Then, it holds that

$$\|\operatorname{div}(A^{\varepsilon} \nabla u_{\varepsilon}^{(2)}) - \operatorname{div}(A^{\operatorname{hom}} \nabla u_0^{\min})\|_{H^{-1}(\Omega)} \leq C \varepsilon^2,$$

for some $C < +\infty$.

Proof. First, note that

$$\operatorname{div}(A^{\varepsilon}(x) \nabla u_{\varepsilon}^{(2)}(x))_{\varepsilon} - A^{\operatorname{hom}} \nabla u_0^{\min}(x) = \sum_{i=1}^N \partial_i \left((A^{\varepsilon}(x) \nabla u_{\varepsilon}^{(2)}(x) - A^{\operatorname{hom}} \nabla u_0^{\min}(x))_i \right),$$

where, for $i = 1, \dots, N$, the i -th component $(A^{\varepsilon}(x) \nabla u_{\varepsilon}^{(2)}(x) - A^{\operatorname{hom}} \nabla u_0^{\min}(x))_i$ is given by

$$(A^{\varepsilon}(x) \nabla u_{\varepsilon}^{(2)}(x) - A^{\operatorname{hom}} \nabla u_0^{\min}(x))_i = P_{i,\varepsilon}(x) + \varepsilon Q_{i,\varepsilon}(x) + \varepsilon^2 R_{i,\varepsilon}(x), \quad (4.3)$$

with

$$\begin{aligned} P_{i,\varepsilon}(x) &:= \sum_{j=1}^N a_{ij}^{\varepsilon}(x) \partial_j u_0^{\min}(x) + \sum_{j,k=1}^N a_{ik}^{\varepsilon}(x) \partial_{y_k} \psi_j^{\varepsilon}(x) \partial_j u_0^{\min}(x) - \sum_{j=1}^N a_{ij}^{\operatorname{hom}} \partial_j u_0^{\min}(x), \\ Q_{i,\varepsilon}(x) &:= \sum_{j,k=1}^N \psi_j^{\varepsilon}(x) a_{ik}^{\varepsilon}(x) \partial_{kj}^2 u_0^{\min}(x) + \sum_{k,j,l=1}^N a_{il}^{\varepsilon}(x) \partial_{y_l} \chi_{kj}^{\varepsilon}(x) \partial_{kj}^2 u_0^{\min}(x) + \sum_{j=1}^N a_{ij}^{\varepsilon}(x) \partial_j \tilde{u}_1(x) \\ &\quad + \sum_{j,k=1}^N a_{ik}^{\varepsilon}(x) \partial_{y_k} \psi_j^{\varepsilon}(x) \partial_j \tilde{u}_1(x), \\ R_{i,\varepsilon}(x) &:= \sum_{k,j,l=1}^N \chi_{kj}^{\varepsilon}(x) a_{il}^{\varepsilon}(x) \partial_{lkj}^3 u_0^{\min}(x) + \sum_{j,k=1}^N \psi_j^{\varepsilon}(x) a_{ik}^{\varepsilon}(x) \partial_{kj}^2 \tilde{u}_1(x). \end{aligned} \quad (4.4)$$

We now estimate the above terms, starting from $P_{i,\varepsilon}(x)$. Note that $P_{i,\varepsilon}(x)$ rewrites as

$$P_{i,\varepsilon}(x) = \sum_{j=1}^N g_i^{j,\varepsilon}(x) \partial_j u_0^{\min}(x), \quad \text{for all } i = 1, \dots, N,$$

where g_i^j is defined as

$$g_i^j(y) := a_{ij}(y) + \sum_{k=1}^N a_{ik}(y) \partial_{y_k} \psi_j(y) - a_{ij}^{\text{hom}}, \quad \text{for all } i, j = 1, \dots, N.$$

For fixed $j = 1, \dots, N$, set $G^j := (g_i^1, \dots, g_i^N)$. We have that $G^j \in L^2_{\text{sol}}(Y)$. Indeed, thanks to problem (3.7) satisfied by ψ_j , it follows that

$$\operatorname{div} G^j(y) = \sum_{i=1}^N \partial_{y_i} g_i^j(y) = 0.$$

Therefore, by applying Proposition 3.2, the components of G^j are represented by

$$g_i^j(y) = \sum_{k=1}^N \partial_{y_k} \alpha_{ik}^j(y). \quad (4.5)$$

Note that $\langle g_i^j \rangle_Y = 0$, since

$$a_{ij}^{\text{hom}} = \left\langle a_{ij}(y) + \sum_{k=1}^N a_{ik}(y) \partial_{y_k} \psi_j(y) \right\rangle_Y. \quad (4.6)$$

Using the representation (4.5) as well as the Leibniz rule, we get that

$$\begin{aligned} P_{i,\varepsilon}(x) &= \sum_{j,k=1}^N \partial_{y_k} \alpha_{ik}^{j,\varepsilon}(x) \partial_j u_0^{\min}(x) \\ &= \varepsilon \left(\sum_{j,k=1}^N \partial_k (\alpha_{ik}^{j,\varepsilon} \partial_j u_0^{\min})(x) - \sum_{j,k=1}^N \alpha_{ik}^{j,\varepsilon}(x) \partial_{kj}^2 u_0^{\min}(x) \right). \end{aligned}$$

Hence, (4.3) turns into

$$(A^\varepsilon(x) \nabla u_\varepsilon^{(2)}(x) - A^{\text{hom}} \nabla u_0^{\min}(x))_i = \varepsilon \left(\sum_{j,k=1}^N \partial_k (\alpha_{ik}^{j,\varepsilon} \partial_j u_0^{\min})(x) + \tilde{Q}_{i,\varepsilon}(x) \right) + \varepsilon^2 R_{i,\varepsilon}(x), \quad (4.7)$$

where

$$\tilde{Q}_{i,\varepsilon}(x) := Q_{i,\varepsilon}(x) - \sum_{j,k=1}^N \alpha_{ik}^{j,\varepsilon}(x) \partial_{kj}^2 u_0^{\min}(x),$$

with $Q_{i,\varepsilon}$ being defined as in (4.4). We now estimate $\tilde{Q}_{i,\varepsilon}(x)$. As for $P_{i,\varepsilon}$, $\tilde{Q}_{i,\varepsilon}(x)$ can be rewritten as

$$\tilde{Q}_{i,\varepsilon}(x) = \sum_{j,k=1}^N h_i^{jk,\varepsilon}(x) \partial_{kj}^2 u_0^{\min}(x) + \sum_{j=1}^N t_i^{j,\varepsilon}(x) \partial_j \tilde{u}_1(x), \quad \text{for all } i = 1, \dots, N, \quad (4.8)$$

where for any fixed $j, k = 1, \dots, N$, the functions h_i^{kj} and t_i^j are defined as

$$h_i^{kj}(y) := \psi_j(y) a_{ik}(y) + \sum_{l=1}^N a_{il}(y) \partial_{y_l} \chi_{kj}(y) - \alpha_{ik}^j(y), \quad (4.9)$$

$$t_i^j(y) := a_{ij}(y) + \sum_{k=1}^N a_{ik}(y) \partial_{y_k} \psi_j(y). \quad (4.10)$$

We claim that $H^{kj} := (h_1^{kj}, \dots, h_N^{kj})$ as well as $T^j := (t_1^j, \dots, t_N^j)$ belong to $L^2_{\text{sol}}(Y)$. Indeed, bearing in mind that χ_{kj}^ε is the solution to (3.9) together with the fact that

$$\sum_{i=1}^N \partial_{y_i}(-\alpha_{ik}^j) = \sum_{i=1}^N \partial_{y_i}(\alpha_{ki}^j) = g_k^j,$$

we deduce that for fixed $j, k = 1, \dots, N$,

$$\begin{aligned} \operatorname{div}(H^{kj}) &= \sum_{i=1}^N \partial_{y_i} h_i^{kj}(y) = \sum_{i=1}^N \partial_{y_i}(\psi_j^\varepsilon a_{ik}^\varepsilon)(y) + \sum_{i=1}^N \partial_{y_i} \left(\sum_{l=1}^N a_{il} \partial_l \chi_{kj} \right)(y) + \sum_{i=1}^N \partial_{y_i}(-\alpha_{ik}^j)(y) \\ &= \sum_{i=1}^N \partial_{y_i}(\psi_j(y) a_{ik}(y)) - b_{kj}(y) + \int_Y b_{kj} dy + g_k^j(y) = 0, \end{aligned}$$

where we have used the equality $a_{kj}^{\text{hom}} = \langle b_{kj} \rangle_Y$ (cf. (3.10) and (4.6)). Likewise, since ψ_j is the solution to (3.7), we immediately conclude that

$$\operatorname{div}(T^j) = \sum_{i=1}^N \partial_{y_i} t_i^j(y) = \sum_{i=1}^N \partial_{y_i} \left(a_{ij}(y) + \sum_{k=1}^N a_{ik}(y) \partial_{y_k} \psi_j(y) \right) = 0.$$

Therefore, applying Proposition 3.2, it follows that the components of H^{kj} and T^j are represented by

$$h_i^{kj}(y) = \langle h_i^{kj} \rangle_Y + \sum_{l=1}^N \partial_{y_l} \beta_{il}^{kj}(y) \quad \text{and} \quad t_i^j(y) = \langle t_i^j \rangle_Y + \sum_{k=1}^N \partial_{y_k} \gamma_{ik}^j(y),$$

for all $i = 1, \dots, N$. This implies that

$$\begin{aligned} \tilde{Q}_{i,\varepsilon}(x) &= \sum_{j,k=1}^N \langle h_i^{kj} \rangle_Y \partial_{kj}^2 u_0^{\min}(x) + \sum_{j,k,l=1}^N \partial_{y_l} \beta_{il}^{kj,\varepsilon}(x) \partial_{kj}^2 u_0^{\min}(x) \\ &\quad + \sum_{j=1}^N \langle t_i^j \rangle_Y \partial_j \tilde{u}_1(x) + \sum_{j,k=1}^N \sum_{l=1}^N \partial_{y_k} \gamma_{ik}^{j,\varepsilon}(x) \partial_j \tilde{u}_1(x) \\ &= \sum_{j,k=1}^N \langle h_i^{kj} \rangle_Y \partial_{kj}^2 u_0^{\min}(x) + \sum_{j=1}^N \langle t_i^j \rangle_Y \partial_j \tilde{u}_1(x) \\ &\quad + \varepsilon \left(\sum_{j,k,l=1}^N \partial_l (\beta_{il}^{kj,\varepsilon} \partial_{kj}^2 u_0^{\min})(x) - \sum_{j,k,l=1}^N \beta_{il}^{kj,\varepsilon}(x) \partial_{lkj}^3 u_0^{\min}(x) \right. \\ &\quad \left. + \sum_{j,k=1}^N \partial_i (\gamma_{ik}^{j,\varepsilon} \partial_j \tilde{u}_1)(x) - \sum_{j,k=1}^N \gamma_{ik}^{j,\varepsilon}(x) \partial_{kj}^2 \tilde{u}_1(x) \right) \end{aligned}$$

From (4.3) together with (4.7), we conclude that

$$\begin{aligned} (A^\varepsilon(x) \nabla u_\varepsilon^{(2)}(x) - A^{\text{hom}} \nabla u_0^{\min}(x))_i &= \varepsilon \left(\sum_{j,k=1}^N \partial_k (\alpha_{ik}^{j,\varepsilon} \partial_j u_0^{\min})(x) + \sum_{j,k=1}^N \langle h_i^{kj} \rangle_Y \partial_{kj}^2 u_0^{\min}(x) + \sum_{j=1}^N \langle t_i^j \rangle_Y \partial_j \tilde{u}_1(x) \right) \\ &\quad + \varepsilon^2 \left(\sum_{j,k,l=1}^N \partial_l (\beta_{il}^{kj,\varepsilon} \partial_{kj}^2 u_0^{\min})(x) + \sum_{j,k=1}^N \partial_i (\gamma_{ik}^{j,\varepsilon} \partial_j \tilde{u}_1)(x) + \tilde{R}_{i,\varepsilon}(x) \right), \end{aligned}$$

where $\tilde{R}_{i,\varepsilon}(x)$ is defined as

$$\tilde{R}_{i,\varepsilon}(x) := R_{i,\varepsilon}(x) - \sum_{j,k,l=1}^N \beta_{il}^{kj,\varepsilon}(x) \partial_{lkj}^3 u_0^{\min}(x) - \sum_{j,k=1}^N \gamma_{ik}^{j,\varepsilon}(x) \partial_{kj}^2 \tilde{u}_1(x).$$

Noticing that $\langle h_i^{kj} \rangle_Y = c_{ijk}$ (cf. (3.12) and (4.9)) and $\langle t_i^j \rangle_Y = a_{ij}^{\text{hom}}$ (cf. (4.6) and (4.10)) as well as bearing in mind problem (3.11), we obtain that

$$\sum_{i=1}^N \partial_i \left(\sum_{j,k=1}^N \langle h_i^{kj} \rangle_Y \partial_{kj}^2 u_0^{\min}(x) + \sum_{j=1}^N \langle t_i^j \rangle_Y \partial_j \tilde{u}_1(x) \right) = 0.$$

Moreover, since $\alpha_{ik}^j = -\alpha_{ki}^j$, it follows that

$$\sum_{i=1}^N \partial_i \left(\sum_{j,k=1}^N \partial_k (\alpha_{ik}^{j,\varepsilon} \partial_j u_0^{\min})(x) \right) = 0.$$

Likewise, since $\beta_{il}^{kj} = -\beta_{li}^{kj}$ and $\gamma_{ik}^j = -\gamma_{ki}^j$,

$$\sum_{i=1}^N \partial_i \left(\sum_{j,k,l=1}^N \partial_l (\beta_{il}^{kj,\varepsilon} \partial_{kj}^2 u_0^{\min})(x) \right) = \sum_{i=1}^N \partial_i \left(\sum_{j,k=1}^N \partial_i (\gamma_{ik}^{j,\varepsilon} \partial_j \tilde{u}_1)(x) \right) = 0.$$

Therefore,

$$\operatorname{div}(A^\varepsilon(x) \nabla u^{(2)}(x)_\varepsilon - A^{\text{hom}} \nabla u_0^{\min}(x)) = \varepsilon^2 \operatorname{div}(\tilde{R}_\varepsilon(x)),$$

with $\tilde{R}_\varepsilon = (\tilde{R}_{1,\varepsilon}, \dots, \tilde{R}_{N,\varepsilon})$. Now,

$$\|\operatorname{div}(A^\varepsilon \nabla u_\varepsilon^{(2)} - A^{\text{hom}} \nabla u_0^{\min})\|_{H^{-1}(\Omega)} = \|\operatorname{div} \tilde{R}_\varepsilon\|_{H^{-1}(\Omega)} \leq \|\tilde{R}_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^2,$$

which concludes the proof. $\square$

5. COMPACTNESS RESULT IN THEOREM 2.7

In this section, we are going to prove a compactness result stated in Theorem 2.7(i).

Proposition 5.1. *Let $\{\varepsilon_n\}_n$ be a sequence such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Suppose that Assumptions (H1)-(H3) hold. If $\{u_n\}_n \subset H^1(\Omega)$ is a sequence such that*

$$\sup_n F_n^1(u_n) < \infty,$$

then, $\{u_n\}_n$ converges to $u_0^{\min} \in H^1(\Omega)$ weakly in $H^1(\Omega)$.

Proof. First, recall that thanks to Assumption (H3), the homogenized matrix A_{hom} satisfies the same growth conditions with the same constants, i.e.,

$$\alpha|\xi|^2 \leq A_{\text{hom}} \xi \cdot \xi \leq \beta|\xi|^2,$$

for all $\xi \in \mathbb{R}^N$. Let $\lambda > 0$ that will be fixed later. Note that

$$-ab \geq -\frac{a^2}{2\lambda^2} - \frac{\lambda^2}{2}b^2, \tag{5.1}$$

for all $a, b \geq 0$. Let $C_\Omega > 0$ be the Poincaré constant of Ω , i.e. such that,

$$\|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_{L^2(\Omega; \mathbb{R}^N)}, \tag{5.2}$$

for all $v \in H_0^1(\Omega)$. Recall that if $u \in L^2(\Omega)$ is such that $F_n(u) < +\infty$, then $u \in H_0^1(\Omega)$. By using (H3), (5.1) and (5.2), we get

$$\begin{aligned} F_n(u) &\geq \alpha \|\nabla u\|_{L^2(\Omega; \mathbb{R}^N)}^2 - \frac{\lambda^2}{2} \|u\|_{L^2(\Omega)}^2 - \frac{1}{2\lambda^2} \|f\|_{L^2(\Omega)}^2 \\ &\geq \left(\alpha - \frac{\lambda^2}{2} C_\Omega^2 \right) \|\nabla u\|_{L^2(\Omega; \mathbb{R}^N)}^2 - \frac{1}{2\lambda^2} \|f\|_{L^2(\Omega)}^2. \end{aligned}$$

Namely,

$$\left(\alpha - \frac{\lambda^2}{2} C_\Omega^2 \right) \|\nabla u\|_{L^2(\Omega; \mathbb{R}^N)}^2 \leq F_n(u).$$

Choosing

$$\lambda \in \left(0, \frac{\sqrt{2\alpha}}{C_\Omega}\right),$$

yields that

$$\|\nabla u\|_{L^2(\Omega; \mathbb{R}^N)} \leq C(F_n(u) + 1) \leq C(F_n^1(u) + 1),$$

where the constant $C > 0$ changes all the times. Since $u \in H_0^1(\Omega)$, using the Poincaré inequality again, we get that

$$\|u\|_{H^1(\Omega)} < C(F_n^1(u) + 1).$$

Therefore, if $\{u_n\}_{n \in \mathbb{N}} \subset L^2(\Omega)$ is such that

$$\sup_{n \in \mathbb{N}} F_n^1(u_n) < +\infty,$$

then, there exists a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ such that $u_{n_k} \rightharpoonup v$ weakly in $H^1(\Omega)$, for some $v \in H^1(\Omega)$.

We now prove that $v = u_0^{\min}$. Assume by contradiction that $v \neq u_0^{\min}$. Since the minimizer $u_0^{\min}$ is unique, it follows that

$$\liminf_{k \rightarrow \infty} F_{n_k}(u_{n_k}) \geq F_{\text{hom}}^0(v) > F_{\text{hom}}^0(u_0^{\min}).$$

Thus,

$$\lim_{k \rightarrow \infty} F_{n_k}^1(u_{n_k}) = \lim_{k \rightarrow \infty} \frac{F_{n_k}(u_{n_k}) - F_{\text{hom}}^0(u_0^{\min})}{\varepsilon_{n_k}} = +\infty.$$

This gives the desired contradiction. Since the limit is unique, we also get that the full sequence converges. $\square$

Remark 5.2. Note that, in order to get compactness, we do not need to have Assumptions (H4) and (H5) in force.

6. THE LIMINF INEQUALITY FOR THEOREM 2.7

In this section, we prove the lower bound of Theorem 2.7 (ii).

Proposition 6.1. *Assume that Assumptions (H1)-(H5) hold. Then, for any sequence $\{u_n\}_n \subset L^2(\Omega)$ converging to $u_0^{\min}$ with respect to $L^2(\Omega)$ it holds that*

$$\liminf_{n \rightarrow \infty} F_n^1(u_n) \geq F_{\text{hom}}^1(u_0^{\min}).$$

Proof. Without loss of generality, we can assume that

$$\liminf_{n \rightarrow \infty} F_n^1(u_n) < \infty.$$

Let $u_n^{(2)}$ be the function defined in (4.2). Then, it holds that

$$\begin{aligned} \liminf_{n \rightarrow \infty} F_n^1(u_n) &= \liminf_{n \rightarrow \infty} \frac{F_n(u_n) - F_{\text{hom}}^0(u_0^{\min})}{\varepsilon_n} \\ &\geq \liminf_{n \rightarrow \infty} \frac{F_n(u_n^{\min}) - F_{\text{hom}}^0(u_0^{\min})}{\varepsilon_n} \\ &\geq \liminf_{n \rightarrow \infty} \frac{F_n(u_n^{\min}) - F_n(u_n^{(2)})}{\varepsilon_n} + \liminf_{n \rightarrow \infty} \frac{F_n(u_n^{(2)}) - F_{\text{hom}}^0(u_0^{\min})}{\varepsilon_n} \\ &= \liminf_{n \rightarrow \infty} I_n^1 + \liminf_{n \rightarrow \infty} I_n^2 + \liminf_{n \rightarrow \infty} I_n^3, \end{aligned}$$

where

$$\begin{aligned} I_n^1 &:= \frac{F_n(u_n^{\min}) - F_n(u_n^{(2)})}{\varepsilon_n}, \\ I_n^2 &:= \frac{1}{\varepsilon_n} \left[\int_{\Omega} A^{\varepsilon_n}(x) \nabla u_n^{(2)}(x) \cdot \nabla u_n^{(2)}(x) \, dx - \int_{\Omega} A_{\text{hom}} u_0^{\min}(x) \cdot u_0^{\min}(x) \, dx \right], \end{aligned}$$

and

$$I_n^3 := \frac{1}{\varepsilon_n} \int_{\Omega} f(x)(u_n^{(2)}(x) - u_0^{\min}(x)) \, dx.$$

We now claim that

$$\lim_{n \rightarrow \infty} I_n^1 = 0,$$

and that

$$\lim_{n \rightarrow \infty} (I_n^2 + I_n^3) = F_{\text{hom}}^1(u_0^{\min}).$$

These will give the desired result.

Step 1: limit of I_n^1 . We have that

$$\begin{aligned} I_n^1 &= \frac{1}{\varepsilon_n} \left[\int_{\Omega} A^{\varepsilon_n}(x) \nabla u_n^{(2)}(x) \cdot \nabla u_n^{(2)}(x) \, dx - \int_{\Omega} A^{\varepsilon_n}(x) \nabla u_n^{\min}(x) \cdot \nabla u_n^{\min}(x) \, dx \right] \\ &\quad - \frac{1}{\varepsilon_n} \int_{\Omega} f(x)(u_n^{(2)}(x) - u_n^{\min}(x)) \, dx. \end{aligned}$$

Writing $u_n^{(2)} = u_n^{\min} + (u_n^{(2)} - u_n^{\min})$ gives

$$\begin{aligned} &\frac{1}{\varepsilon_n} \left| \int_{\Omega} A^{\varepsilon_n}(x) \nabla u_n^{(2)}(x) \cdot \nabla u_n^{(2)}(x) \, dx - \int_{\Omega} A^{\varepsilon_n}(x) \nabla u_n^{\min}(x) \cdot \nabla u_n^{\min}(x) \, dx \right| \\ &= \frac{1}{\varepsilon_n} \left| \int_{\Omega} A^{\varepsilon_n}(x) \nabla(u_n^{(2)} - u_n^{\min})(x) \cdot \nabla(u_n^{(2)} - u_n^{\min})(x) \, dx \right. \\ &\quad \left. + \frac{2}{\varepsilon_n} \int_{\Omega} A^{\varepsilon_n}(x) \nabla(u_n^{(2)} - u_n^{\min})(x) \cdot \nabla u_n^{(2)}(x) \, dx \right| \\ &\leq \frac{1}{\varepsilon_n} \left\| \operatorname{div} \left(A^{\varepsilon_n} \nabla(u_n^{(2)} - u_n^{\min}) \right) \right\|_{H^{-1}(\Omega)} \left(\|u_n^{(2)} - u_n^{\min}\|_{H_0^1(\Omega)} + 2\|u_n^{(2)}\|_{H_0^1(\Omega)} \right) \\ &\leq \varepsilon_n \left(\|u_n^{\min}\|_{H_0^1(\Omega)} + 3\|u_n^{(2)}\|_{H_0^1(\Omega)} \right), \end{aligned} \tag{6.1}$$

where the last step follows from Proposition 4.9. Note that from (H3), together with the fact that $u_n^{(2)} - u_n^{\min} \in H_0^1(\Omega)$, it holds that

$$\begin{aligned} \alpha \|\nabla(u_n^{(2)} - u_n^{\min})\|_{L^2(\Omega)}^2 &\leq \int_{\Omega} A^{\varepsilon_n}(x) \nabla(u_n^{(2)} - u_n^{\min})(x) \cdot \nabla(u_n^{(2)} - u_n^{\min})(x) \, dx \\ &\leq \sup_{\varphi \in H_0^1(\Omega)} \int_{\Omega} A^{\varepsilon_n}(x) \nabla(u_n^{(2)} - u_n^{\min})(x) \cdot \nabla \varphi(x) \, dx \\ &= \left\| \operatorname{div} \left(A^{\varepsilon_n} \nabla(u_n^{(2)} - u_n^{\min}) \right) \right\|_{H^{-1}(\Omega)} \|u_n^{(2)} - u_n^{\min}\|_{H_0^1(\Omega)} \\ &\leq (1 + C_{\Omega}) \left\| \operatorname{div} \left(A^{\varepsilon_n} \nabla(u_n^{(2)} - u_n^{\min}) \right) \right\|_{H^{-1}(\Omega)} \|\nabla(u_n^{(2)} - u_n^{\min})\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, calling C_{Ω} the Poincaré constant of Ω , we obtain that

$$\begin{aligned} \left| \frac{1}{\varepsilon_n} \int_{\Omega} f(x)(u_n^{(2)}(x) - u_n^{\min}(x)) \, dx \right| &\leq \frac{1}{\varepsilon_n} \|f\|_{L^2(\Omega)} \|u_n^{(2)} - u_n^{\min}\|_{L^2(\Omega)} \\ &\leq \frac{C_{\Omega}}{\varepsilon_n} \|f\|_{L^2(\Omega)} \|\nabla(u_n^{(2)} - u_n^{\min})\|_{L^2(\Omega)} \\ &\leq \frac{(1 + C_{\Omega})C_{\Omega}}{\alpha \varepsilon_n} \|f\|_{L^2(\Omega)} \left\| \operatorname{div} \left(A^{\varepsilon_n} \nabla(u_n^{(2)} - u_n^{\min}) \right) \right\|_{H^{-1}(\Omega)} \\ &\leq \frac{(1 + C_{\Omega})C_{\Omega}}{\alpha} \|f\|_{L^2(\Omega)} \varepsilon_n, \end{aligned} \tag{6.2}$$

where the last step follows by Proposition 4.9. Since

$$\sup_n \left(\|u_n^{\min}\|_{H_0^1(\Omega)} + 3\|u_n^{(2)}\|_{H_0^1(\Omega)} \right) < \infty,$$

from (6.1) and (6.2), we get the desired result.

Step 2: limit of I_n^2 . Note that

$$\begin{aligned} \nabla u_{\varepsilon_n}^{(2)}(x) &= \nabla \left(u_0^{\min}(x) + \varepsilon_n \sum_{i=1}^N \psi_i^{\varepsilon_n}(x) \partial_i u_0^{\min}(x) + \varepsilon_n^2 \sum_{r,s=1}^N \chi_{rs}^{\varepsilon_n}(x) \partial_{rs}^2 u_0^{\min}(x) - \varepsilon_n^2 \sum_{l=1}^N \psi_l^{\varepsilon_n}(x) \partial_l \tilde{u}_1(x) \right) \\ &= \nabla u_0^{\min}(x) + \sum_{i=1}^n \nabla_y \psi_i^{\varepsilon_n}(x) \partial_i u_0^{\min}(x) \\ &\quad + \varepsilon_n \left[\sum_{i=1}^N \psi_i^{\varepsilon_n}(x) \nabla(\partial_i u_0^{\min}(x)) + \sum_{r,s=1}^N \nabla_y(\chi_{rs}^{\varepsilon_n}(x)) \partial_{rs}^2 u_0^{\min}(x) - \sum_{l=1}^N \nabla_y(\psi_l^{\varepsilon_n}(x)) \partial_l \tilde{u}_1(x) \right] \\ &\quad + o(\varepsilon_n). \end{aligned}$$

Therefore,

$$\int_{\Omega} A^{\varepsilon_n}(x) \nabla u_{\varepsilon_n}^{(2)}(x) \cdot \nabla u_{\varepsilon_n}^{(2)}(x) dx = H_n + \varepsilon_n G_n + o(\varepsilon_n),$$

where H_{ε_n} and G_{ε_n} are defined as

$$\begin{aligned} H_n &:= \int_{\Omega} A^{\varepsilon_n}(x) \left(\nabla u_0^{\min}(x) + \sum_{i=1}^N \nabla_y(\psi_i^{\varepsilon_n}(x)) \partial_i u_0^{\min}(x) \right) \cdot \\ &\quad \cdot \left(\nabla u_0^{\min}(x) + \sum_{i=1}^n \nabla_y(\psi_i^{\varepsilon_n}(x)) \partial_i u_0^{\min}(x) \right) dx, \end{aligned}$$

and

$$\begin{aligned} G_n &:= 2 \int_{\Omega} A^{\varepsilon_n}(x) \left(\nabla u_0^{\min}(x) + \sum_{i=1}^N \nabla_y(\psi_i^{\varepsilon_n}(x)) \partial_i u_0^{\min}(x) \right) \cdot \left(\sum_{i=1}^N \psi_i^{\varepsilon_n}(x) \nabla(\partial_i u_0^{\min}(x)) \right. \\ &\quad \left. + \sum_{r,s=1}^N \nabla_y(\chi_{rs}^{\varepsilon_n}(x)) \partial_{rs}^2 u_0^{\min}(x) - \sum_{l=1}^N \nabla_y(\psi_l^{\varepsilon_n}(x)) \partial_l \tilde{u}_1(x) \right) dx, \end{aligned}$$

respectively.

Step 2.1: asymptotic behavior of H_n . Using Proposition 4.3, and recalling that we are assuming the barycenter of Y to be at the origin, we get that

$$\begin{aligned} &\int_{\Omega} A^{\varepsilon_n}(x) \nabla u_0^{\min}(x) \cdot \nabla u_0^{\min}(x) dx \\ &= \sum_{i,j=1}^N \int_{\Omega} a_{ij}^{\varepsilon_n}(x) \partial_i u_0^{\min}(x) \partial_j u_0^{\min}(x) dx \\ &= \sum_{i,j=1}^N \langle a_{ij} \rangle_Y \int_{\Omega} \partial_i u_0^{\min}(x) \partial_j u_0^{\min}(x) dx \\ &\quad + \varepsilon_n \sum_{i,j=1}^N \int_{\Omega} \nabla(\partial_i u_0^{\min} \partial_j u_0^{\min})(x) dx \cdot \langle y a_{ij} \rangle_Y + o(\varepsilon_n). \end{aligned} \tag{6.3}$$

Moreover,

$$\int_{\Omega} A^{\varepsilon_n}(x) \left(\sum_{i=1}^N \nabla_y(\psi_i^{\varepsilon_n}(x)) \partial_i u_0^{\min}(x) \right) \cdot \nabla u_0^{\min}(x) dx$$

$$\begin{aligned}
&= \sum_{i,j,s=1}^N \int_{\Omega} a_{js}^{\varepsilon_n}(x) \partial_{y_s} \psi_i^{\varepsilon_n}(x) \partial_i u_0^{\min}(x) \partial_j u_0^{\min}(x) \, dx \\
&= \sum_{i,j,s=1}^N \langle a_{js} \partial_{y_s} \psi_i \rangle_Y \int_{\Omega} \partial_i u_0^{\min}(x) \partial_j u_0^{\min}(x) \, dx \\
&\quad + \varepsilon_n \sum_{i,j,s=1}^N \int_{\Omega} \nabla(\partial_i u_0^{\min} \partial_j u_0^{\min})(x) \, dx \cdot \langle y a_{js} \partial_{y_s} \psi_i \rangle_Y + o(\varepsilon_n). \tag{6.4}
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_{\Omega} A^{\varepsilon_n}(x) \left(\sum_{i=1}^N \nabla_y(\psi_i^{\varepsilon_n}(x)) \partial_i u_0^{\min}(x) \right) \cdot \left(\sum_{j=1}^N \nabla_y(\psi_j^{\varepsilon_n}(x)) \partial_j u_0^{\min}(x) \right) \, dx \\
&= \sum_{i,j,r,s=1}^N \int_{\Omega} a_{rs}^{\varepsilon_n}(x) \partial_{y_s} \psi_i^{\varepsilon_n}(x) \partial_i u_0^{\min}(x) \partial_{y_r} \psi_j^{\varepsilon_n}(x) \partial_j u_0^{\min}(x) \, dx \\
&= \sum_{i,j,r,s=1}^N \langle a_{rs} \partial_{y_s} \psi_i \partial_{y_r} \psi_j \rangle_Y \int_{\Omega} \partial_i u_0^{\min}(x) \partial_j u_0^{\min}(x) \, dx \\
&\quad + \varepsilon_n \sum_{i,j,r,s=1}^N \int_{\Omega} \nabla(\partial_i u_0^{\min} \partial_j u_0^{\min})(x) \, dx \cdot \langle y a_{rs} \partial_{y_s} \psi_i \partial_{y_r} \psi_j \rangle_Y + o(\varepsilon_n). \tag{6.5}
\end{aligned}$$

Therefore, from (6.3), (6.4), (6.5), and the definition of A_{hom} (see (2.1)) we obtain that

$$\begin{aligned}
H_n &= \int_{\Omega} A_{\text{hom}} \nabla u_0^{\min}(x) \cdot \nabla u_0^{\min}(x) \, dx \\
&\quad + \varepsilon_n \sum_{i,j=1}^N \int_{\Omega} \nabla(\partial_i u_0^{\min} \partial_j u_0^{\min})(x) \, dx \cdot \langle y [a_{ij} + 2Ae_j \cdot \nabla \psi_i + A \nabla \psi_i \cdot \nabla \psi_j] \rangle_Y + o(\varepsilon_n). \tag{6.6}
\end{aligned}$$

Step 2.2: limit of G_n . Therefore,

$$\lim_{n \rightarrow \infty} \frac{F_n(u_n^{\min}) - F_{\text{hom}}^0(u_0^{\min})}{\varepsilon_n} = \liminf_{n \rightarrow \infty} G_{\varepsilon_n}(u_n^{\min})$$

To make the computations more clear, we split $G_n(u_n^{\min}) := G_n(u_0^{\min}, \tilde{u}_1)$ as follows

$$\begin{aligned}
G_n(u_0^{\min}, \tilde{u}_1) &:= G_n^1(u_0^{\min}, \tilde{u}_1) + G_n^2(u_0^{\min}, \tilde{u}_1) + G_n^3(u_0^{\min}, \tilde{u}_1) + G_n^4(u_0^{\min}, \tilde{u}_1) \\
&\quad + G_n^5(u_0^{\min}, \tilde{u}_1) + G_n^6(u_0^{\min}, \tilde{u}_1),
\end{aligned}$$

where the functionals $G_n^i(u_0^{\min}, \tilde{u}_1)$, for $i = 1, \dots, 6$ are given by

$$\begin{aligned}
G_n^1(u_0^{\min}, \tilde{u}_1) &:= 2 \int_{\Omega} A^{\varepsilon_n}(x) \nabla u_0^{\min}(x) \cdot \sum_{i=1}^N \psi_i^{\varepsilon_n}(x) \nabla(\partial_i u_0^{\min}(x)) \, dx, \\
G_n^2(u_0^{\min}, \tilde{u}_1) &:= 2 \int_{\Omega} A^{\varepsilon_n}(x) \nabla u_0^{\min}(x) \cdot \sum_{r,s=1}^N \nabla_y(\chi_{rs}^{\varepsilon_n}(x)) \partial_{rs}^2 u_0^{\min}(x) \, dx, \\
G_n^3(u_0^{\min}, \tilde{u}_1) &:= -2 \int_{\Omega} A^{\varepsilon_n}(x) \nabla u_0^{\min}(x) \cdot \sum_{l=1}^N \nabla_y(\psi_l^{\varepsilon_n}(x)) \partial_l \tilde{u}_1(x) \, dx,
\end{aligned}$$

$$\begin{aligned}
G_n^4(u_0^{\min}, \tilde{u}_1) &:= 2 \int_{\Omega} A^{\varepsilon_n}(x) \sum_{j=1}^N \nabla_y(\psi_j^{\varepsilon_n}(x)) \partial_j u_0^{\min}(x) \cdot \sum_{i=1}^N \psi_i^{\varepsilon_n}(x) \nabla(\partial_i u_0^{\min}) \, dx, \\
G_n^5(u_0^{\min}, \tilde{u}_1) &:= 2 \int_{\Omega} A^{\varepsilon_n}(x) \sum_{j=1}^N \nabla_y(\psi_j^{\varepsilon_n}(x)) \partial_j u_0^{\min}(x) \cdot \sum_{r,s=1}^N \nabla_y(\chi_{rs}^{\varepsilon_n}(x)) \partial_{rs}^2 u_0^{\min}(x) \, dx, \\
G_n^6(u_0^{\min}, \tilde{u}_1) &:= -2 \int_{\Omega} A^{\varepsilon_n}(x) \sum_{j=1}^N \nabla(\psi_j^{\varepsilon_n}(x)) \partial_j u_0^{\min}(x) \cdot \sum_{l=1}^N \nabla_y(\psi_l^{\varepsilon_n}(x)) \partial_l \tilde{u}_1(x) \, dx.
\end{aligned}$$

Now, we separately compute the limit as $n \rightarrow \infty$ of each functionals G_n^i . Using the Riemann-Lebesgue lemma (see Lemma 4.1), we get

$$\begin{aligned}
\lim_{n \rightarrow \infty} G_n^1(u_0^{\min}, \tilde{u}_1) &= 2 \lim_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} \psi_i^{\varepsilon_n}(x) A^{\varepsilon_n}(x) \nabla u_0^{\min}(x) \cdot \partial_i \nabla u_0^{\min}(x) \, dx \\
&= 2 \sum_{i=1}^N \int_{\Omega} \langle \psi_i A \rangle_Y \nabla u_0^{\min}(x) \cdot \partial_i \nabla u_0^{\min}(x) \, dx
\end{aligned} \tag{6.7}$$

Regarding the second functional G_n^2 , we obtain that

$$\begin{aligned}
\lim_{n \rightarrow \infty} G_n^2(u_0^{\min}, \tilde{u}_1) &= 2 \lim_{n \rightarrow \infty} \sum_{r,s=1}^N \int_{\Omega} \nabla u_0^{\min}(x) \cdot A^{\varepsilon_n}(x) \nabla_y(\chi_{rs}^{\varepsilon_n}(x)) \partial_{rs}^2 u_0^{\min}(x) \, dx \\
&= 2 \lim_{n \rightarrow \infty} \sum_{r,s=1}^N \sum_{j=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) (\nabla_y(\chi_{rs}^{\varepsilon_n}(x)) \cdot A^{\varepsilon_n}(x) e_j) \partial_{rs}^2 u_0^{\min}(x) \, dx \\
&= 2 \sum_{r,s=1}^N \sum_{j=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \langle A e_j \cdot \nabla \chi_{rs} \rangle_Y \partial_{rs}^2 u_0^{\min}(x) \, dx.
\end{aligned} \tag{6.8}$$

Here, we have exploited the symmetry of the matrix A to deduce that

$$\begin{aligned}
\nabla u_0^{\min}(x) \cdot A^{\varepsilon_n}(x) \nabla_y(\chi_{rs}^{\varepsilon_n}(x)) &= \sum_{j=1}^N \partial_j u_0^{\min}(x) (A^{\varepsilon_n}(x) \nabla_y \chi_{rs}^{\varepsilon_n}(x))_j \\
&= \sum_{j=1}^N \partial_j u_0^{\min}(x) (A^{\varepsilon_n}(x) \nabla_y \chi_{rs}^{\varepsilon_n}(x) \cdot e_j) \\
&= \sum_{j=1}^N \partial_j u_0^{\min}(x) (\nabla_y \chi_{rs}^{\varepsilon_n}(x) \cdot A^{\varepsilon_n}(x) e_j).
\end{aligned}$$

Similarly for G_n^3 , it follows that

$$\begin{aligned}
\lim_{n \rightarrow \infty} G_n^3(u_0^{\min}, \tilde{u}_1) &= -2 \lim_{n \rightarrow \infty} \sum_{l=1}^N \int_{\Omega} \nabla u_0^{\min}(x) \cdot A^{\varepsilon_n}(x) \nabla_y(\psi_l^{\varepsilon_n}(x)) \partial_l \tilde{u}_1(x) \, dx \\
&= -2 \lim_{n \rightarrow \infty} \sum_{l=1}^N \sum_{j=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \cdot [A^{\varepsilon_n}(x) e_j \cdot \nabla_y(\psi_l^{\varepsilon_n}(x))] \partial_l \tilde{u}_1(x) \, dx \\
&= -2 \sum_{l=1}^N \sum_{j=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \langle A e_j \cdot \nabla_y \psi_l \rangle_Y \partial_l \tilde{u}_1(x) \, dx.
\end{aligned} \tag{6.9}$$

The limit $G_n^4(u_0^{\min}, \tilde{u}_1)$ as $n \rightarrow \infty$ reads as follows

$$\begin{aligned} \lim_{n \rightarrow \infty} G_n^4(u_0^{\min}, \tilde{u}_1) &= 2 \lim_{n \rightarrow \infty} \sum_{j=1}^N \sum_{i=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \psi_i^{\varepsilon_n}(x) A^{\varepsilon_n}(x) \nabla_y(\psi_j^{\varepsilon_n}(x)) \cdot \partial_i(\nabla u_0^{\min}(x)) \, dx \\ &= 2 \sum_{j=1}^N \sum_{i=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \langle \psi_i A \nabla_y \psi_j \rangle_Y \cdot \partial_i(\nabla u_0^{\min}(x)) \, dx. \end{aligned} \quad (6.10)$$

For G_n^5 , it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} G_n^5(u_0^{\min}, \tilde{u}_1) &= 2 \lim_{n \rightarrow \infty} \sum_{j=1}^N \sum_{r,s=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) A^{\varepsilon_n}(x) \nabla_y(\psi_j^{\varepsilon_n}(x)) \cdot \nabla_y(\chi_{rs}^{\varepsilon_n}(x)) \partial_{rs}^2 u_0^{\min}(x) \, dx \\ &= 2 \sum_{j=1}^N \sum_{r,s=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \langle A \nabla_y \psi_j \cdot \nabla_y \chi_{rs} \rangle_Y \partial_{rs}^2 u_0^{\min}(x) \, dx. \end{aligned}$$

The variational formulation of the problem of the corrector ψ_j (see (3.7)) with χ_{rs} as a test function yields

$$\langle A \nabla_y(\psi_j) \cdot \nabla_y(\chi_{rs}) \rangle_Y = \int_Y A(y) \nabla_y \psi_j(y) \cdot \nabla_y \chi_{rs}(y) \, dy = - \int_Y A(y) e_j \cdot \nabla_y \chi_{rs}(y) \, dy = - \langle A e_j \cdot \nabla_y \chi_{rs} \rangle_Y.$$

This implies that

$$\lim_{n \rightarrow \infty} G_n^5(u_0^{\min}, \tilde{u}_1) = -2 \sum_{j=1}^N \sum_{r,s=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \langle A e_j \cdot \nabla_y \chi_{rs} \rangle_Y \partial_{rs}^2 u_0^{\min}(x) \, dx. \quad (6.11)$$

Finally, the limit of G_n^6 as $n \rightarrow \infty$ is

$$\begin{aligned} \lim_{n \rightarrow \infty} G_n^6(u_0^{\min}, \tilde{u}_1) &= -2 \lim_{n \rightarrow \infty} \sum_{j=1}^N \sum_{l=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) A^{\varepsilon_n}(x) \nabla_y(\psi_j^{\varepsilon_n}(x)) \cdot \nabla_y(\psi_l^{\varepsilon_n}(x)) \partial_l \tilde{u}_1(x) \, dx \\ &= -2 \sum_{j=1}^N \sum_{l=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \langle A \nabla_y \psi_j \cdot \nabla_y \psi_l \rangle_Y \partial_l \tilde{u}_1(x) \, dx. \end{aligned}$$

Using again the variational formulation of the problem of the corrector ψ_j (see (3.7)), choosing as test function ψ_l , yields to

$$\langle A \nabla_y \psi_j \cdot \nabla_y \psi_l \rangle_Y = - \langle A e_j \cdot \nabla_y \psi_l \rangle_Y.$$

Thanks to this equality, it follows that

$$\lim_{n \rightarrow \infty} G_n^6(u_0^{\min}, \tilde{u}_1) = 2 \sum_{j=1}^N \sum_{l=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \langle A e_j \cdot \nabla_y \psi_l \rangle_Y \partial_l \tilde{u}_1(x) \, dx. \quad (6.12)$$

Gathering formulas (6.7)-(6.12) and noting that (6.8) and (6.11) as well as (6.9) and (6.12) cancel out, we deduce that

$$\begin{aligned} \lim_{n \rightarrow \infty} G_n(u_0^{\min}, \tilde{u}_1) &= \lim_{n \rightarrow \infty} G_n^1(u_0^{\min}, \tilde{u}_1) + \lim_{n \rightarrow \infty} G_n^4(u_0^{\min}, \tilde{u}_1) \\ &= 2 \sum_{i=1}^N \int_{\Omega} \langle \psi_i A \rangle_Y \nabla u_0^{\min}(x) \cdot \partial_i(u_0^{\min}(x)) \, dx + 2 \sum_{j=1}^N \sum_{i=1}^N \int_{\Omega} \partial_j u_0^{\min}(x) \langle \psi_i A \nabla_y \psi_j \rangle_Y \cdot \partial_i(\nabla u_0^{\min}(x)) \, dx. \end{aligned} \quad (6.13)$$

Step 3: limit of I_n^3 . A direct computation shows that

$$\lim_{n \rightarrow \infty} \frac{1}{\varepsilon_n} \int_{\Omega} f(x) (u_n^{(2)}(x) - u_0^{\min}(x)) \, dx = \lim_{n \rightarrow \infty} \sum_{j=1}^N \int_{\Omega} f(x) \psi_j^{\varepsilon_n}(x) \partial_j u_0^{\min}(x) \, dx$$

$$= \sum_{j=1}^N \langle \psi_j \rangle_Y \int_{\Omega} f(x) \partial_j u_0^{\min}(x) \, dx. \quad (6.14)$$

□

7. THE LIMSUP INEQUALITY FOR THEOREM 2.7

In this section, we prove the upper bound of Theorem 2.7 (ii).

Proposition 7.1. *Assume Assumptions (H1)-(H5) to hold. Then, there exists a sequence $\{u_n\}_n \subset H_0^1(\Omega)$ converging to $u_0^{\min}$ weakly in $H^1(\Omega)$ such that*

$$\lim_{n \rightarrow \infty} F_n^1(u_n) = F_{\text{hom}}^1(u_0^{\min}).$$

Proof. We define, for $n \in \mathbb{N}$,

$$u_n(x) := u_0^{\min}(x) + \varepsilon \sum_{i=1}^N \psi_i^{\varepsilon_n}(x) \partial_i u_0^{\min}(x).$$

The weak convergence of $\{u_n\}_n$ to $u_0^{\min}$ is a direct computation. Moreover, the desired convergence of the energy follows from (6.7), (6.9), (6.10), and (6.14). □

8. PROOF OF THEOREM 2.12

This section is devoted to the proof of Theorem 2.12. We first illustrate the idea of the proof by considering the special case

$$V(x, p) = a(x)|p|^2,$$

where $a : \mathbb{R}^N \rightarrow \mathbb{R}$ is an Y -periodic function such that $0 < c_1 \leq a(c) \leq c_2 < +\infty$ for all $x \in Y$. Then, it holds that

$$V_{\text{hom}}(p) = \min \left\{ \int_Y a(y) |p + \varphi(y)|^2 \, dy : \varphi \in L^2(Y), \int_Y \varphi(y) \, dy = 0 \right\}.$$

This minimization problem admits a solution, that we can compute explicitly. Indeed, the Euler-Lagrange equation gives the existence of a constant $c(p) \in \mathbb{R}$ such that

$$2a(y)(p + \varphi(y)) = c(p),$$

for almost every $y \in Y$. Recalling that $a(y) > 0$, we obtain

$$\varphi(y) = \frac{c(p)}{2a(y)} - p.$$

Moreover, the constant $c(p)$ can be computed by the zero average requirement for φ :

$$c(p) = 2p \left(\int_Y \frac{1}{a(y)} \, dy \right)^{-1}.$$

Thus,

$$V_{\text{hom}}(p) = p^2 \left(\int_Y \frac{1}{a(y)} \, dy \right)^{-1},$$

which yields

$$G_{\text{hom}}(v) = \int_{\Omega} v^2(x) \, dx \left(\int_Y \frac{1}{a(y)} \, dy \right)^{-1},$$

for all $v \in L^2(\Omega)$.

Fix $m \in \mathbb{R}$. We now consider the solution $v_0^{\min} \in L^2(\Omega)$ to the minimization problem

$$\min \left\{ G_{\text{hom}}(v) : v \in L^2(\Omega), \int_{\Omega} v(x) \, dx = m \right\}.$$

The Euler-Lagrange equation gives the existence of a constant $c \in \mathbb{R}$ such that

$$2v_0^{\min}(x) \left(\int_Y \frac{1}{a(y)} dy \right)^{-1} = c,$$

for almost every $x \in \Omega$. Thus, $v_0^{\min}$ is constant. The mass constraint gives that

$$v_0^{\min}(x) = \frac{m}{|\Omega|},$$

for all $x \in \Omega$, which yields

$$G_{\text{hom}}(v_0^{\min}) = \frac{m^2}{|\Omega|} \left(\int_Y \frac{1}{a(y)} dy \right)^{-1}. \quad (8.1)$$

In a similar way, we can obtain the solution $v_n^{\min}$ to the minimization problem

$$\min \left\{ G_{\varepsilon_n}(v) : v \in L^2(\Omega), \int_{\Omega} v(x) dx = m \right\}.$$

This gives

$$v_n^{\min}(x) = \frac{c_n}{2a^{\varepsilon_n}(x)},$$

where

$$c_n := 2m \left(\int_{\Omega} \frac{1}{a^{\varepsilon_n}(x)} dx \right)^{-1}.$$

Thus,

$$G_n(v_n^{\min}) = m^2 \left(\int_{\Omega} \frac{1}{a^{\varepsilon_n}(x)} dx \right)^{-1}. \quad (8.2)$$

Note that thanks to our assumptions on Ω and ε_n , we get

$$\int_{\Omega} \frac{1}{a^{\varepsilon_n}(x)} dx = |\Omega| \int_Y \frac{1}{a(y)} dy.$$

Therefore, from (8.1) and (8.2), we conclude that

$$G_n(v_n^{\min}) = G_{\text{hom}}(v_0^{\min}),$$

for all $n \in \mathbb{N} \setminus \{0\}$, as desired.

In the general case, we use a similar argument as the one implemented above. We first consider the homogenized energy density

$$V_{\text{hom}}(p) = \min \left\{ \int_Y V(y, p + \varphi(y)) dy : \varphi \in L^2(Y), \int_Y \varphi(y) dy = 0 \right\}.$$

Since we are assuming $p \mapsto V(y, p)$ to be strictly convex, for each $y \in Y$ we denote by $\partial_p V^{-1}(y) : \mathbb{R} \rightarrow \mathbb{R}$ the inverse of the map $\partial_p V(y, \cdot)$, and we use the notation $v \mapsto \partial_p^{-1} V(y)[v]$ to avoid the use of too many round parenthesis. We get that the optimal perturbation φ satisfies

$$\varphi(y) = \partial_p^{-1} V(y)[c(p)] - p,$$

for some $c(p) \in \mathbb{R}$, which can be computed by using the zero average constrained

$$p = \int_Y \partial_p^{-1} V(y)[c(p)] dy.$$

If we now consider the solution $v_0^{\min} \in L^2(\Omega)$ to the minimization problem

$$\min \left\{ G_{\text{hom}}(v) : v \in L^2(\Omega), \int_{\Omega} v(x) dx = m \right\},$$

we get that

$$v_0^{\min} = \frac{m}{|\Omega|}.$$

In particular,

$$G_{\text{hom}}(v_0^{\min}) = |\Omega| \int_Y V \left(y, \partial_p^{-1} V(y) \left[c \left(\frac{m}{|\Omega|} \right) \right] \right) dy.$$

If we now consider the solution $v_n^{\min}$ to the minimization problem

$$\min \left\{ G_n(v) : v \in L^2(\Omega), \int_{\Omega} v(x) dx = m \right\},$$

we get that

$$\partial_p V \left(\frac{x}{\varepsilon_n}, v_n^{\min}(x) \right) = c_n,$$

where, using the mass constraint,

$$m = \int_{\Omega} \partial_p^{-1} V \left(\frac{x}{\varepsilon_n} \right) [c_n] dx = |\Omega| \int_Y \partial_p^{-1} V(y) [c_n] dy.$$

This gives that

$$G_n(v_n^{\min}) = \int_{\Omega} V \left(\frac{x}{\varepsilon_n}, \partial_p^{-1} V \left(\frac{x}{\varepsilon_n} \right) [c_n] \right) dx = |\Omega| \int_Y V(y, \partial_p^{-1} V(y) [c_n]) dy.$$

It is possible to choose

$$c_n = \frac{m}{|\Omega|},$$

which gives $G_n(v_n^{\min}) = G_{\text{hom}}(v_0^{\min})$ for all $n \in \mathbb{N} \setminus \{0\}$ as desired.

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